THERMOCONVECTIVE WAVES IN A LAYER WITH FREE BOUNDARIES

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We show that in a thin layer of a liquid, for which the influence of its boundaries is substantial, the propagation of weakly attenuating thermoconvective waves is possible close to and beyond the stability threshold. We obtain the characteristics of weakly attenuating harmonics.

In a previous paper [1] it was shown that in a horizontal layer of a thermally compressible liquid, with a temperature gradient parallel to the graviational force (we assume that the liquid expands when heated, i.e., that $\partial\rho/\partial T < 0$; in the contrary case, we select the opposite direction for the temperature gradient), the propagation of weakly attenuating thermoconvective waves is possible. The authors have studied the propagation of small one-dimensional perturbations of the temperature, velocity, and pressure under the assumption that the horizontal boundaries alone stipulate the mechanical equilibrium of the layer and in no way affect the propagation of the waves, i.e., that the condition $\lambda/h \ll 1$ is satisfied, where λ is the wave length of the waves in question and h is the layer thickness. Such an assumption can apparently be made only for a thick layer. We have not excluded the possibility that, in principle, it is not realizable. At least in thin layers a substantial influence of the boundaries on the characteristics of the waves should be expected.

It is possible to have weakly attenuating thermoconvective waves propagating in a thin layer of a liquid? In order to answer this question we need to study the propagation of thermoconvective waves with the influence of the boundaries taken into account.

We consider a horizontal semiinfinite layer of a liquid, bounded above and below by two planes between which a constant temperature difference is maintained. We assume that $\partial\rho/\partial T < 0$, and that the temperature gradient $\vec{\gamma}$ is parallel to the gravitational force $\vec{g}$. On a vertical wall bounding the liquid from the side we have a source of periodic perturbations of the temperature, velocity, and pressure, the nature of which we do not specify here. The perturbations propagate along the layer in the form of thermoconvective waves,

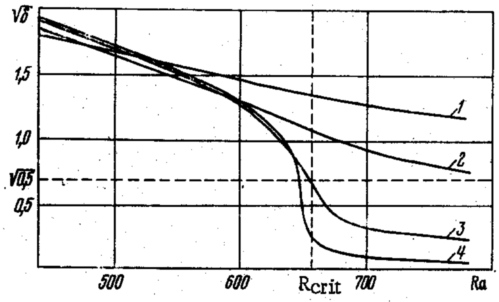

Fig. 1

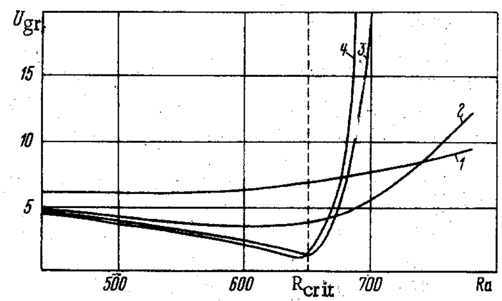

Fig. 2

Fig. 1. Attenuation decrement versus Rayleigh number (Pr = 1): Curves 1, 2, 3, and 4 are for ω equal to 3, 1, 0.1, and 0.01, respectively.

Fig. 2. Group velocity versus Rayleigh number (Pr = 1): Curves 1, 2, 3, and 4 are for ω equal to 3, 1, 0.1, and 0.01, respectively.

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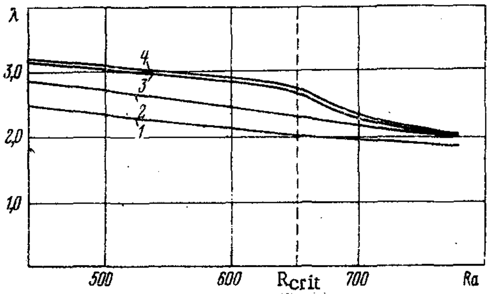

Fig. 3. Wave length versus Rayleigh number (Pr = 1): Curves 1, 2, 3, and 4 are for ω equal to 3, 1, 0.1, and 0.01, respectively.

the characteristics of which are determined by the quantity γ, the thickness h, the properties of the liquid, and the boundary conditions. It is known that in the case $\gamma = 0$ (the liquid is isothermal) the thermoconvective waves are strongly damped, the logarithmic damping decrement $\delta > 2\pi$, which corresponds to an exp (2π)-fold damping per wave length.

We investigate how the characteristics of the thermoconvective waves vary with an increase in the value of the temperature gradient γ. It should be expected that for a given layer there is a critical value γ_{crit} for which there occurs a principal change in the nature of the thermoconvective waves, if only because of the fact that there is a critical value γ_c beyond which even the mechanical equilibrium of the layer is unstable [2]. Thus there is a stability threshold of thermoconvective waves, which is determined by both the properties of the layer and the character of the periodic perturbations on the boundary (the amplitude, the frequency, the shape). It is necessary to take into account the fact that beyond the stability threshold the amplitude of flows in the layer is not determined only by the amplitude of the perturbations on the side wall.

To answer the question concerning the possibility of the propagation of weakly attenuating waves in the layer, we investigate waves up to and beyond the stability threshold. We formulate the problem mathematically. We choose a cartesian system of coordinates. The x axis is directed along the layer, the y axis normal to it. Our starting point is from the well known equations of natural convection in the Boussinesq approximation [2]. We consider only waves of small amplitude. This enables us to neglect the convective terms in the equations; furthermore, we can expect that the stability threshold of thermoconvective waves of small amplitude will be close to the stability threshold of the layer, a fact which is well known. By virtue of the fact that in an above-critical region the amplitude of the convection arising is proportional to $(\Delta Ra/Ra)^{1/2}$, we can assume that linear equations will give a good description of the process, even in a neighborhood beyond the stability threshold of the waves [3]. Consequently, for the study of thermoconvective waves in the layer of liquid it is necessary to solve the following dimensionless system of linear equations:

$$\begin{gathered}\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \Delta u,\\ \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y} + \Delta v + \mathrm{Gr}\theta,\\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,\\ \frac{\partial \theta}{\partial t} = \frac{1}{\mathrm{Pr}}\Delta\theta + v.\end{gathered} \tag{1}$$

Here, u and v are the velocity components in the direction of the x and y axes, respectively; $T = 1 - y + \theta$ is the temperature; p is the pressure; $\mathrm{Gr} = \beta h^4\gamma/\nu^2$ is the Grashof number; and $\mathrm{Pr} = \nu/a$ is the Prandtl number.

The boundary conditions on the side wall are

$$x = 0;\quad \theta = \theta_0 \sin \pi y \exp(i\omega t);\quad v = v_0 \sin \pi y \exp(i\omega t). \tag{2}$$

Horizontal boundaries; we consider three versions here:

a) two solid planes;

b) one solid plane and one free surface;

c) two free surfaces.

The last case is characterized by the simplicity of the formulas and is considered in the present paper. For this case we have

$$y = 0,\ 1; \quad \theta = 0; \quad v = 0; \quad \partial u/\partial y = 0. \tag{3}$$

We seek a solution of the equations (1)-(3) in the form

$$(u,\ p) = (U,\ \Pi) \cos \pi y \exp [i(\omega t - kx)],$$
$$(v,\ \theta) = (V,\ \Theta) \sin \pi y \exp [i(\omega t - kx)]. \tag{4}$$

Here U, V, Π, Θ are constants; $k = k_1 + ik_2$ is a wave vector. After substituting the equations (4) into the equations (1), we obtain the following equation defining the relationship between the wave vector k and the frequency ω:

$$[i\omega + (k^2 + \pi^2)/\mathrm{Pr}]\,(i\omega + k^2 + \pi^2)\,(k^2 + \pi^2) = k^2\,\mathrm{Gr}. \tag{5}$$

The damping decrement, as is well known, may be expressed in terms of the components of the wave vector as follows: $\delta = 2\pi k_2/k_1$, and, as can be seen from equation (5), it depends on the frequency, the Prandtl number, and the Grashof number. To answer the question as to whether we can have propagation of waves with weak damping, and, if so, for what parameter values, we need to solve equation (5) for k. Equation (5) has 6 roots, three of which can be rejected because of the condition $k_2 < 0$. From the roots which remain we must select the one to which the minimum damping decrement corresponds. The entire procedure of solving equation (5) and selecting an appropriate k was carried out on an electronic digital computer.

Figure 1 shows how the attenuation decrement δ depends on the Rayleigh number Ra for various values of ω. The point $Ra = R_{crit}$ corresponds to the threshold of stability of the mechanical equilibrium of the layer with free boundaries. Under our assumptions the Rayleigh number Ra, corresponding to the stability threshold of thermoconvective waves generated by periodic perturbations on the side wall, differs little from R_{crit}.

An analysis of the graphs shows that with an increase in the Rayleigh number Ra, the attenuation decrement decreases for all frequencies, i.e., the presence of a negative temperature gradient across the layer leads to a weakening of the damping of the thermoconvective waves. However, the form of this dependence differs substantially for the various frequencies close to the critical number R_{crit}. For sufficiently large ω (Curves 1, 2) there is no essential change in the attenuation decrement as Ra increases, even when passing through the stability threshold. The picture is completely different in the case of small ω (Curves 3, 4). With an increase in Ra the attenuation decrement varies up to the stability threshold as for large ω, but close to $Ra = R_{crit}$ it decreases substantially. Harmonics appear corresponding to the weak damping of the thermoconvective waves (Curve 4). The region of weak damping is determined by Rayleigh numbers larger than some critical value R_{crit} and by sufficiently small value of ω. For weakly damped waves the propagation speed of perturbations increases sharply (Fig. 2). The wave lengths, as can be seen from Fig. 3, vary between the limits of 2h and 3h. This indicates that the boundaries have a substantial influence on the propagation of thermoconvective waves.

We may thus conclude that in a thin layer of a liquid the propagation of weakly attenuating thermoconvective waves is possible in the presence of a temperature gradient parallel to the gravitational force, but only close to and beyond the stability threshold.

We give numerical estimates for a layer of air of centimeter thickness: $h = 10^{-2}$ m, $\nu = 15 \cdot 10^{-6}$ m²/sec, $a = 27 \cdot 10^{-6}$ m²/sec. Weak attenuation, can be seen from Fig. 1 ($\delta \simeq 10^{-2}$), is possible for $\omega = 10^2$, Ra = 700. In terms of dimensional quantities this corresponds to a frequency of $\omega = 10^{-3}$ Hz and $\gamma = 300°/\mathrm{m}$. Under these conditions a perturbation attenuates 2.73-fold over 100 wave lengths, i.e., in two meters. We note that in the absence of a vertical temperature gradient a perturbation in such a layer is attenuated by as much as half the height of the layer, i.e., by approximately $5 \cdot 10^{-3}$ m.

The weak attenuation beyond the stability threshold can be explained if we take into account the fact that after the breakdown of mechanical equilibrium a periodic convective structure develops. For the case in which the perturbations on the side wall are, in some sense, commensurate with the period and the intensity of this convection, we can expect them to be weakly attenuated.

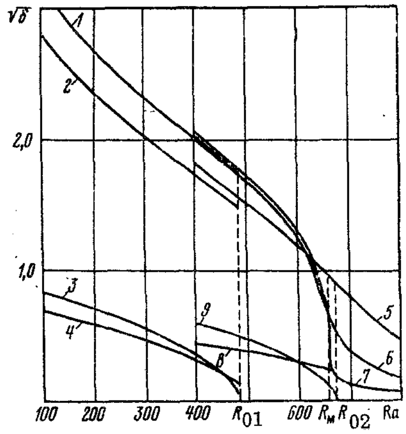

Fig. 4. Attenuation decrement.

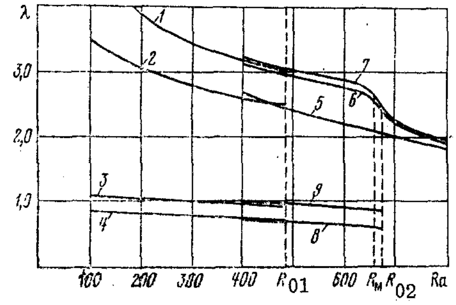

Fig. 5. Wave length.

Having shown that in a layer of liquid, heated from below, weak attenuation of thermoconvective waves is possible close to and beyond the stability threshold, we find it of interest to study the behavior of such waves in viscoelastic media under the same conditions. It is a known fact (see [4]) that in a layer of viscoelastic liquid, heated from below, two kinds of instability are possible: monotonic and oscillatory. It should therefore be expected that in such liquids the process of thermoconvective wave propagation would be a richer one.

To elucidate the fundamental features of the propagation of thermoconvective waves in viscoelastic media, we use the Maxwell model of a viscoelastic liquid, the rheological equation for which, taking incompressibility into account, has the form

$$\tau_r \frac{\partial \sigma_{kj}}{\partial t} + \sigma_{kj} = \eta \left(\frac{\partial v_k}{\partial x_j} + \frac{\partial v_j}{\partial x_k} \right). \tag{6}$$

Here the σ_{ki} are the components of the viscous stress tensor; τ_r is the characteristic relaxation time; and v_i are the velocity components.

Thus, just as we did above, we limit ourselves to the consideration of waves of small amplitude in a layer with free boundaries.

The dimensionless equations for a viscoelastic medium can be written in the Boussinesq approximation in the form

$$\begin{gathered}
\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial \sigma_{11}}{\partial x} + \frac{\partial \sigma_{12}}{\partial y}, \\
\frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y} + \frac{\partial \sigma_{12}}{\partial x} + \frac{\partial \sigma_{22}}{\partial y} + \mathrm{Gr}\,\theta, \\
\tau \frac{\partial \sigma_{11}}{\partial t} + \sigma_{11} = 2\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \\
\tau \frac{\partial \sigma_{12}}{\partial t} + \sigma_{12} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \frac{\partial \theta}{\partial t} = \frac{1}{\mathrm{Pr}} \Delta\theta + v, \\
\tau \frac{\partial \sigma_{22}}{\partial t} + \sigma_{22} = 2\frac{\partial v}{\partial y}.
\end{gathered} \tag{7}$$

The boundary conditions remain as before, namely, the conditions (2) and (3). The convective motion in the layer of the viscoelastic liquid is determined by four dimensionless parameters: Gr, τ, Pr, ω. The parameter τ characterizes the elastic properties of the medium. When $\tau = 0$, the system (7) describes an ordinary Newtonian liquid.

As before, we seek a solution of the equations (7), (2), and (3) in the form

$$\begin{gathered}
(u, p, \sigma_{11}, \sigma_{22}) = (U, \Pi, \Sigma_{11}, \Sigma_{22}) \cos \pi y \exp[i(\omega t - kx)], \\
(v, \theta, \sigma_{12}) = (V, \Theta, \Sigma_{12}) \sin \pi y \exp[i(\omega t - kx)].
\end{gathered} \tag{8}$$

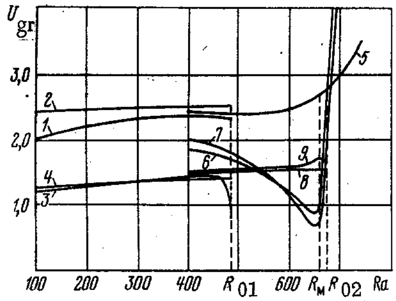

Fig. 6. Group velocity.

For ω and k we obtain the dispersion equation

$$i\omega\,(i\omega\tau+1)\,(k^2+\pi^2)\,[i\omega+(k^2+\pi^2)/\mathrm{Pr}] + (k^2+\pi^2)^2\,[i\omega+(k^2+\pi^2)/\mathrm{Pr}] = k^2\,\mathrm{Gr}\,(i\omega\tau+1). \qquad (9)$$

This equation is valid in a region of Rayleigh numbers not exceeding a threshold value for which instability commences. This is connected with the fact that in the region beyond the threshold the amplitude of the convective motion in the layer is no longer determined by the amplitude of perturbations on the side wall. A detailed study of oscillatory and monotonic instability in a layer of a viscoelastic liquid was given in [4]. We merely note here that for an increase in the Rayleigh number as a function of τ three possibilities arise: the monotonic instability threshold may be below, may coincide with, or it may be above the oscillatory instability threshold. For the case in which $\tau \to 0$, i.e., when the liquid is weakly elastic, the first of these possibilities obtains. When τ is sufficiently large, the crisis of instability is associated with oscillatory perturbations. We can expect that in a layer of a viscoelastic liquid, with a vertical temperature gradient present, two types of weakly attenuating thermoconvective waves can propagate: the one type is present close to the monotonic instability threshold and the second type is present close to the oscillatory instability threshold; in the third possibility, simultaneous propagation of both types of waves is possible.

Figures 4, 5, and 6 show the calculated characteristics of thermoconvective waves in a layer of a viscoelastic liquid as a function of Ra. The Curves 1 to 4 correspond to $\tau = 0.6$, Pr = 10, and $\omega = 0.1$; 1; 7.6; and 10. As Ra increases, the oscillating instability for these values of the parameters appears for Ra = R_{kl} before the appearance of the monotonic instability (Ra = R_m). The frequency of the neutral oscillations is $\omega = 7.6$. From Fig. 4 we see that waves having frequencies close to the frequency of the neutral oscillations (Curves 3, 4) have small attenuation decrements; waves with small frequency are strongly damped (Curves 1, 2). The lengths of the weakly attenuating waves are found to be close to the height of the layer (Fig. 5). The group velocities show almost no change with a change in Ra (Fig. 6).

Curves 5-9 correspond to $\tau = 0.5$, Pr = 10, $\omega = 1$; 0.1; 0.001; 15; 8.3. In this case, as Ra increases, a monotonic instability first appears for Ra = R_M; however, the threshold of the oscillating instability is found in the immediate vicinity beyond the threshold of the monotonic instability (Ra = R_{02}). From Fig. 4 we see that under such conditions a weak attenuation of the waves with frequencies close to 0 is possible (Curve 7), just as in the case of an ordinary liquid, and also with frequencies close to the frequency of the neutral oscillations, $\omega = 8.3$ (Curves 8, 9). The lengths and group velocities of the low frequency weakly attenuating waves behave just as in an ordinary liquid. The lengths of the weakly attenuating waves with frequencies close to the neutral frequency turn out to be two times smaller than the lengths of the low frequency waves (Fig. 5). The group velocities remain almost constant (Fig. 6).

We present numerical estimates. If we take a viscoelastic liquid with the parameters $\tau_g = 1$ sec, $\nu = 10^{-3}$ m^2/sec, $\beta = 10^{-3}$ deg^{-1}, h = 0.014 m, and $\Delta T = 60°$, then weak attenuation is possible for the frequency $\omega = 5 \cdot 10^{-3}$ and 5 Hz.

It should be remarked that at present no viscoelastic liquids are known with parameters for which the oscillating instability appears before the monotonic instability. However, the rapid strides being made in the polymer industry make it appear hopeful that such liquids will be synthesized in the near future. It will then be possible to observe the propagation of weakly attenuating thermoconvective waves of high frequencies, predicted earlier in our paper.

NOTATION

ω is the frequency of fluctuations;
λ is the wavelength;
h is the thickness of liquid layer;
g is the gravity force;
γ is the vertical temperature gradient;
ρ is the fluid density;
β is the thermal expansion coefficient;
ν is the kinematic viscosity;

η	is the dynamic viscosity;
a	is the thermal diffusivity;
T	is the temperature;
Ra	is the Rayleigh number;
$\Delta Ra = Ra - R_{crit}$, R_{crit}	are the critical Rayleigh numbers.

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